Propagation of the Electric Impulse in a Cylindrical Conductor of Limited Length During the Variable Period

The following derivation is from Bright, C. (1898). *Submarine Telegraphs: Their History, Construction and Working*. London. pp. 528–531. Only very minor changes in notation have been made.

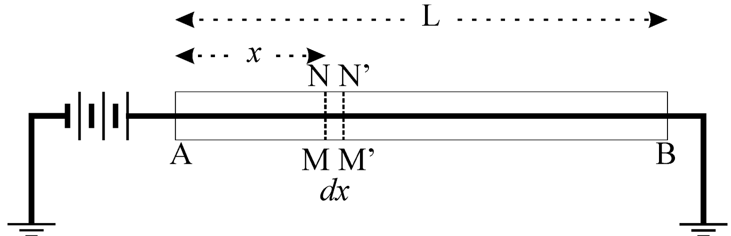

Figure 1:

Let AB (Figure 1) represent a cable with its end B to earth, connected up at A to a battery whose opposite pole is also to earth. In this cable we will consider a given volume of length dx included between two sections at right angles to the axis MN and $M'N'$, and situated at distances $AM = x$ and $AM' = x + dx$ from the point A. Expressing by

V_a the potential at A,

V the potential at M at the time t,

I_a the intensity, or strength, of the current at A at the same instant of time,

I the intensity, or strength, of the current at M at the same instant of time,

I_1 the intensity, or strength, of the current at B at the same instant of time,

ρ the conductor resistance per unit of length,

k the electro-static capacity of the core per unit of length,

r the resistance of the dielectric per unit of length,

L the length AB of the line,

the potential and intensity, or strength, of current at the time t in the section M′ N′ will be $V + dV$ and $I + dI$.

The quantity of electricity Idt which spreads itself over the section MN in the time dt will divide into three parts; the first, which will proceed along the conductor and spread over the section $M'N'$, is represented by $(I + dI)dt$; the second part, which traverses the insulation of the included portion $MN\ M'N'$, whose resistance is $\frac{r}{dx}$, will be equal to

$$\frac{V}{\frac{r}{dx}}dt$$

the third part will increase the electro-static charge in the included portion of cable, and can be expressed by

$$kd\frac{dV}{dt}dt$$

We now have

$$Idt = (I + dI)dt + V\frac{dx}{r}dt + k\frac{dV}{dt}dxdt$$

Reducing and remarking that by Ohm's law

$$I = \frac{dV}{\rho dx} \tag{1}$$

whence

$$dI = -\frac{1}{\rho}\frac{d^2V}{dx^2}dx$$

we get

$$\frac{1}{\rho}\frac{d^2V}{dx^2} - k\frac{dV}{dt} - \frac{1}{r}V = 0$$

or, assuming that

$$k\rho = a^2 \tag{2}$$

and

$$\frac{\rho}{r} = \beta^2 \tag{3}$$

$$\frac{d^2V}{dx^2} - a^2\frac{dV}{dt} - \beta^2 V = 0 \tag{4}$$

The general integration of this equation has been given by Fourier, and is expressed thus–

$$\frac{V}{V_0} = \frac{e^{\beta(L-x)} - e^{-\beta(L-x)}}{e^{\beta L} - e^{-\beta L}} - 2\pi e^{-\frac{t}{kr}} \sum_{n=1}^{n=\infty} \frac{n}{n^2\pi^2 + \beta^2 L^2} e^{-\frac{n^2\pi^2}{a^2L^2}t} \sin\frac{n\pi}{L}x$$

As a rule ρ does not exceed 10 or 12 ohms, and r reaches from 8,000 to 10,000 megohms; β^2 being therefore less than $\frac{1}{10^9}$. If we take $\beta^2 = 0$, which is the same thing as making $r = \infty$, or entirely neglecting loss of electricity through the insulation, the above integration is simplified and becomes

$$\frac{V}{V_0} = \frac{L-x}{L} - 2\sum_{n=1}^{n=\infty} \frac{1}{n\pi} e^{-\frac{n^2\pi^2}{a^2L^2}t} \sin\frac{n\pi}{L}x \tag{5}$$

Differentiating as regards x, and carrying the value obtained for $\frac{dV}{dx}$ into equation (1), we get for the current strength at the distance x

$$I = \frac{V_a}{\rho L}(1 + 2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2L^2}t} \cos\frac{n\pi}{L}x)$$

At the end B of the conductor which is then to earth, $x = 1$; $\sin n\pi$ being always *nil*, the potential is *nil*, and the current I_1 is

$$I_1 = \frac{V_0}{\rho L}(1 + 2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2L^2}t} \cos n\pi) \tag{6}$$

By giving to n consecutive values 1, 2, 3 ... the cosine acquires alternate values equal to -1 and +1. So that if we assume, for abbreviation, that

$$e^{-\frac{\pi^2}{a^2L^2}t} = u \tag{7}$$

equation (6) becomes

$$I_1 = \frac{V_0}{\rho L}[1 - 2(u - u^4 + u^9 - u^{16} + u^{25} - ...)] = F(t) \tag{8}$$

With extremely small values of t, u tends towards unity; at the limit the series $u - u^4 + u^9 ...$ equals $\frac{1}{2}$, and the current intensity is nothing. As the time interval increases, so u diminishes, the series decreasing and the current increasing; but according to Sir William Thomson, the series only differs sensibly from its minimum value $\frac{1}{2}$ when

$$u > \frac{3}{4}$$

Using τ to express the time when this condition is attained, we have

$$e^{-\frac{\pi^2 \tau}{a^2 L^2}} = \frac{3}{4} \tag{9}$$

whence

$$\tau = \frac{a^2 L^2}{\pi^2} \ln \frac{4}{3} \tag{10}$$

being expressed in seconds if a and L are expressed in C.G.S. units,* or

$$\tau = \frac{k \rho L^2}{10^6} \times 0.02915 \text{ second} \tag{11}$$

where k stands for the electro-static capacity of the cable in microfarads per naut, ρ the conductor resistance per naut in ohms, and L its length in nauts,

$$\tau = \frac{RC}{10^6} \times 0.02915 \text{ second} \tag{12}$$

R representing the total conductor resistance in ohms, and K the total capacity of the line in microfarads.

From this point onwards the series tends towards 0, and I_1 increases up to its limit of value $\frac{V_0}{\rho L}$, which is only reached after an infinitely great interval of time.